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Regression Modelling with the Generalized Power Weibull Distribution under Progressive Censoring

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Abstract

The generalized power Weibull (GPW) distribution has recently attracted attention as a flexible model for lifetime data, but existing work has focused mainly on unconditional inference under progressive Type II censoring, comparing maximum likelihood, maximum product spacing and Bayesian estimation in the two-parameter case without covariates. In practical reliability and survival studies, however, lifetimes are typically influenced by explanatory variables and censoring is often implemented through progressive schemes. In this paper we develop a regression framework based on a scale-extended GPW distribution under right censoring, with particular emphasis on progressive Type II designs: the individual-specific scale parameter is linked to covariates via a log-linear model, while the shape parameters are common across units. We derive the likelihood under non-informative censoring, obtain maximum likelihood estimators and their asymptotic covariance matrix, propose maximum product spacing estimators based on transformed uniform variables, and develop a Bayesian version with gamma priors on the shape parameters and a multivariate normal prior on the regression coefficients, estimated via a Metropolis–Hastings-within-Gibbs sampler. A Monte Carlo study examines finite-sample performance across sample sizes, censoring levels and covariate configurations, and a real data analysis with covariates shows that the GPW regression model can outperform classical Weibull and log-normal regression, extending the scope of the GPW family from unconditional modelling to a flexible regression tool for censored lifetime data.

Keywords: generalized power Weibull distribution, lifetime regression, progressive Type II censoring, maximum likelihood, maximum product spacing, Bayesian inference

1. Introduction

The analysis of lifetime and reliability data plays a central role in engineering, biomedical sciences and industrial quality control [1, 2]. Parametric models are particularly attractive when they provide flexible hazard shapes and interpretable parameters, and when they can be embedded into regression frameworks that incorporate covariate information. Among the classical models, the Weibull distribution has long been the workhorse for lifetime analysis [3], but its monotone hazard function is often too restrictive for modern datasets exhibiting non-monotone failure behaviour. This has motivated the development of numerous generalizations of the Weibull family with additional shape parameters and richer hazard structures [4, 5].

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The generalized power Weibull (GPW) distribution was introduced as one such extension, able to generate a wide variety of density and hazard shapes through two shape parameters [6]. It has been shown to provide good fits in several applications, and its analytical tractability allows explicit forms for the density, distribution and quantile functions. More recently, [7] studied parameter estimation for the GPW distribution under progressive Type II censoring, comparing maximum likelihood (ML), maximum product spacing (MPS) and Bayesian estimators in the unconditional two-parameter setting and suggesting optimal censoring schemes based on mean square error and relative efficiency criteria.

However, in many practical situations the lifetimes of interest are strongly influenced by explanatory variables such as operating conditions, treatment group, environmental stress or design characteristics. In the reliability context, one typically wishes to quantify how covariates modify the distribution of component lifetimes, for example through accelerated failure time or proportional hazards structures [2, 8, 9]. In such cases, purely unconditional models are insufficient, and one needs parametric regression models that can handle covariates and censoring simultaneously. While regression models based on classical distributions such as the Weibull, log-normal or log-logistic have been extensively studied [1, 2], there appears to be no regression framework based on the GPW distribution, despite its attractive flexibility.

The aim of this paper is to fill this gap by developing a regression model based on a scale-extended GPW distribution under right censoring, with special attention to progressive Type II censoring schemes [10]. We proceed as follows. First, we recall the basic properties of the GPW distribution and extend it by introducing a positive scale parameter; this leads to a three-parameter GPW family that nests the standard version used in earlier work. Second, we define a log-linear regression structure for the individual-specific scale parameter, allowing lifetimes to depend on a vector of covariates in a way reminiscent of accelerated failure time models. Third, we derive the likelihood under independent right censoring and show that progressive Type II censoring can be accommodated within this framework because it yields non-informative right-censoring times.

We then address parameter estimation using three different approaches. The first is maximum likelihood estimation, for which we derive the log-likelihood and score equations and discuss numerical optimization and asymptotic inference based on the observed information matrix. The second is maximum product spacing, an estimation method originally proposed for complete data and later extended to censored samples; it often provides estimators with similar efficiency to ML while being more robust to certain model misspecifications [11, 12]. In the regression setting, we formulate MPS estimators by transforming observed lifetimes to the uniform scale via the conditional distribution functions and maximizing the geometric mean of the spacings between ordered transformed values. The third approach is Bayesian, in which we assign independent gamma priors to the positive shape parameters and a multivariate normal prior to the regression coefficients. Posterior inference is implemented using a Metropolis–Hastings-within-Gibbs algorithm which alternately updates the regression vector and the shape parameters, and credible intervals are obtained from the posterior draws [13, 14].

To assess the performance of the proposed estimators, we conduct a Monte Carlo simulation study under several scenarios. We vary the sample size, the degree of progressive censoring, and the configuration of covariates, and we evaluate the estimators in terms of bias, root mean squared error, coverage probabilities and empirical average lengths of confidence or credible intervals. The simulation results suggest that the ML and MPS estimators perform comparably in most settings, with MPS being slightly more stable in small samples with heavy censoring, while the Bayesian estimators enjoy improved robustness at high censoring levels, provided that the priors are not strongly misspecified.

Finally, we illustrate the practical usefulness of the GPW regression model through an application to a real reliability dataset involving component lifetimes measured under varying operating conditions. We compare the proposed model to standard Weibull and log-normal regression in terms of goodness-of-fit, predictive accuracy and estimated covariate effects. The GPW regression model provides a better description of the data and yields interpretable covariate effects that would be obscured under simpler parametric forms.

The rest of the paper is organized as follows. Section 2 introduces the GPW distribution, its scale extension and the regression structure. Section 3 describes progressive Type II censoring and derives the likelihood under right censoring. Section 4 presents the ML, MPS and Bayesian estimation methods. Section 5 reports the results of the simulation study, and Section 6 provides the real data analysis. Concluding remarks and directions for future research are given in Section 7.

2. The generalized power Weibull regression model

2.1. The generalized power Weibull distribution

Let X be a positive random variable. Following [7], we say that X has a generalized power Weibull (GPW) distribution with parameters $\theta > 0$ and $\alpha > 0$ if its cumulative distribution function (cdf) is given by

$$F_0(x;\theta,\alpha) \;=\; 1-\exp\Big(1-(1+x^{\alpha})^{\theta}\Big), \qquad x>0. \tag{1}$$

The corresponding probability density function (pdf) is

$$f_0(x;\theta,\alpha) \;=\; \theta\alpha x^{\alpha-1}(1+x^{\alpha})^{\theta-1}\exp\Big(1-(1+x^{\alpha})^{\theta}\Big), \qquad x>0. \tag{2}$$

The survival function $S_0(x) = 1 - F_0(x)$ is therefore

$$S_0(x;\theta,\alpha) \;=\; \exp\Big(1-(1+x^{\alpha})^{\theta}\Big), \qquad x>0. \tag{3}$$

From (2) and (3), the hazard rate function is

$$h_0(x;\theta,\alpha) \;=\; \frac{f_0(x;\theta,\alpha)}{S_0(x;\theta,\alpha)} \;=\; \theta\alpha x^{\alpha-1}(1+x^{\alpha})^{\theta-1}, \qquad x>0. \tag{4}$$

Depending on the values of (θ,α), the GPW distribution can exhibit decreasing, increasing, bathtub-shaped or unimodal hazard functions, providing substantially more flexibility than the classical Weibull model. Explicit expressions for the quantile function and moments can be derived, and random variates can be generated by inversion.

2.2. Scale extension

For regression modelling it is convenient to include a separate scale parameter. We therefore define a three-parameter GPW distribution by introducing a positive scale parameter $\lambda > 0$ and setting

$$X = \lambda T, \qquad T \sim \mathrm{GPW}(\theta,\alpha) \tag{5}$$

in the sense of the standard two-parameter GPW distribution with unit scale as in (1). The resulting cdf for X is

$$F(x;\theta,\alpha,\lambda) \;=\; 1-\exp\Big(1-(1+(x/\lambda)^{\alpha})^{\theta}\Big), \qquad x>0, \tag{6}$$

with

$$f(x;\theta,\alpha,\lambda) \;=\; \frac{\theta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1}\exp\Big(1-(1+(x/\lambda)^{\alpha})^{\theta}\Big), \qquad x>0. \tag{7}$$

The survival and hazard functions are

$$S(x;\theta,\alpha,\lambda) \;=\; \exp\Big(1-(1+(x/\lambda)^{\alpha})^{\theta}\Big), \tag{8}$$

$$h(x;\theta,\alpha,\lambda) \;=\; \frac{\theta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{\alpha}\right)^{\theta-1}. \tag{9}$$

The original two-parameter distribution is recovered by fixing $\lambda = 1$. In the extended model, λ can be interpreted as a scale or acceleration parameter: larger values of λ correspond to stochastically larger lifetimes.

2.3. *Scale regression structure*

Suppose we observe lifetime data on n units, together with a p-dimensional vector of covariates $\boldsymbol{z}_i$ for unit i. We assume that, conditionally on the covariates, the lifetimes X_i are independent and follow a GPW distribution with common shape parameters $\theta > 0$ and $\alpha > 0$, but with unit-specific scale parameters $\lambda_i > 0$ linked to the covariates through a log-linear model,

$$\log\lambda_i = \boldsymbol{z}_i^{\top}\boldsymbol{\beta}, \qquad i=1,\ldots,n, \tag{10}$$

where $\boldsymbol{\beta}\in\mathbb{R}^p$ is a vector of regression coefficients. Equivalently,

$$\lambda_i = \exp(\boldsymbol{z}_i^{\top}\boldsymbol{\beta}), \qquad i=1,\ldots,n. \tag{11}$$

Conditional on $\boldsymbol{z}_i$, the cdf, pdf, survival and hazard functions of X_i are obtained by replacing λ with λ_i in (6)–(9). For example,

$$F_i(x) \;=\; 1-\exp\Big(1-(1+(x/\lambda_i)^{\alpha})^{\theta}\Big), \qquad \lambda_i=\exp(\boldsymbol{z}_i^{\top}\boldsymbol{\beta}). \tag{12}$$

Model (10) is analogous to an accelerated failure time model: the random variable $T_i = X_i/\lambda_i$ has a common GPW(θ,α) distribution across units, while λ_i controls the stretching or compression of time for unit i. Positive values of a coefficient β_k indicate that increasing the corresponding covariate z_{ik} leads to larger values of λ_i, and therefore longer lifetimes, whereas negative coefficients are associated with shorter lifetimes.

3. Progressive censoring and likelihood formulation

3.1. *Progressive Type II censoring*

In many life-testing experiments, practical constraints make it impossible to observe all units until failure. A widely-used design is progressive Type II censoring. Consider an initial sample of n identical units placed on test at time zero. Let $X_{1:n} < \cdots < X_{n:n}$ denote the ordered failure times. Under a progressive Type II censoring scheme with m observed failures and censoring vector $(R_1, \ldots, R_m)$, the experiment proceeds as follows. When the first failure occurs at time $X_{1:n}$, R_1 surviving units are randomly removed from the test. When the second failure occurs, R_2 surviving units are removed, and so on, until the m-th failure at time $X_{m:n}$, when the remaining

$$R_m = n - m - \sum_{j=1}^{m-1} R_j,$$

units are withdrawn. Thus only the first m failure times are observed, and the remaining units are right-censored at the time of the last observed failure.

Under the usual assumption that the removal of units at each failure time is random and independent of their unobserved potential failure times, progressive Type II censoring can be viewed as a particular form of non-informative right censoring. In a regression context where individual covariates are recorded for all units before the start of the test, one can regard the observed data as consisting of subject-specific triplets $(t_i, \delta_i, \boldsymbol{z}_i)$, where t_i is the observed lifetime or censoring time for unit i, δ_i is the failure indicator, and $\boldsymbol{z}_i$ is the covariate vector. The precise ordering or labelling of the units does not affect the likelihood as long as the censoring mechanism remains non-informative.

3.2. Likelihood under right censoring

Let $(T_i, \Delta_i, \boldsymbol{Z}_i)$, $i = 1, \ldots, n$, denote the observed data, where T_i is the observed time, Δ_i is the indicator of failure ($\Delta_i = 1$) or right censoring ($\Delta_i = 0$), and $\boldsymbol{Z}_i$ is the covariate vector. Under the GPW regression model with parameters $\boldsymbol{\beta}$, θ and α, the conditional pdf and survival functions for T_i given $\boldsymbol{Z}_i$ are

$$f(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha) = f(T_i; \theta, \alpha, \lambda_i), \qquad S(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha) = S(T_i; \theta, \alpha, \lambda_i), \tag{13}$$

with $\lambda_i = \exp(\boldsymbol{Z}_i^\top \boldsymbol{\beta})$ and $f(\cdot)$, $S(\cdot)$ given by (7)–(8). Assuming independent units and non-informative right censoring, the likelihood function is

$$L(\boldsymbol{\beta}, \theta, \alpha) \;=\; \prod_{i=1}^{n} \Big[f(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha)\Big]^{\Delta_i} \Big[S(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha)\Big]^{1-\Delta_i}. \tag{14}$$

Taking logarithms, the log-likelihood becomes

$$\ell(\boldsymbol{\beta}, \theta, \alpha) \;=\; \sum_{i=1}^{n} \left\{\Delta_i \log f(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha) + (1 - \Delta_i) \log S(T_i \mid \boldsymbol{Z}_i; \boldsymbol{\beta}, \theta, \alpha)\right\}. \tag{15}$$

For the GPW regression model this can be written explicitly as

$$\ell(\boldsymbol{\beta}, \theta, \alpha) = \sum_{i=1}^{n} \Delta_i \Bigg[\log(\theta\alpha) - \log\lambda_i + (\alpha - 1)\log\frac{T_i}{\lambda_i} + (\theta - 1)\log\Big(1 + (T_i/\lambda_i)^{\alpha}\Big) + 1 - (1 + (T_i/\lambda_i)^{\alpha})^{\theta}\Bigg]$$
$$+ \sum_{i=1}^{n} (1 - \Delta_i)\Big[1 - (1 + (T_i/\lambda_i)^{\alpha})^{\theta}\Big], \tag{16}$$

where $\lambda_i = \exp(\boldsymbol{Z}_i^\top \boldsymbol{\beta})$. This is the basic object used for maximum likelihood and Bayesian estimation.

4. Estimation methods

4.1. Maximum likelihood estimation

Maximum likelihood estimators (MLEs) of $(\boldsymbol{\beta}, \theta, \alpha)$ are obtained by solving the score equations obtained from (16). Let

$$\boldsymbol{\eta}_i = \boldsymbol{Z}_i^\top \boldsymbol{\beta}, \qquad \lambda_i = \exp(\boldsymbol{\eta}_i), \qquad y_i = T_i/\lambda_i, \qquad u_i = 1 + y_i^\alpha.$$

Then (16) can be written more compactly as

$$\ell(\boldsymbol{\beta}, \theta, \alpha) = \sum_{i=1}^n \Delta_i[\log(\theta\alpha) - \boldsymbol{\eta}_i + (\alpha - 1)\log y_i + (\theta - 1)\log u_i + 1 - u_i^\theta] + \sum_{i=1}^n (1 - \Delta_i)[1 - u_i^\theta]. \tag{17}$$

Differentiating with respect to $\boldsymbol{\beta}$, θ and α yields the score functions

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \sum_{i=1}^n \left\{ \Delta_i \left[-\boldsymbol{Z}_i + (\alpha - 1)\frac{\partial \log y_i}{\partial \boldsymbol{\beta}} + (\theta - 1)\frac{1}{u_i}\frac{\partial u_i}{\partial \boldsymbol{\beta}} - \theta u_i^{\theta-1}\frac{\partial u_i}{\partial \boldsymbol{\beta}} \right] + (1 - \Delta_i)\left[-\theta u_i^{\theta-1}\frac{\partial u_i}{\partial \boldsymbol{\beta}} \right] \right\}, \tag{18}$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^n \left\{ \Delta_i \left[\frac{1}{\theta} + \log u_i - u_i^\theta \log u_i \right] + (1 - \Delta_i)[-u_i^\theta \log u_i] \right\}, \tag{19}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^n \left\{ \Delta_i \left[\frac{1}{\alpha} + \log y_i + (\theta - 1)\frac{y_i^\alpha \log y_i}{u_i} - \theta u_i^{\theta-1} y_i^\alpha \log y_i \right] + (1 - \Delta_i)\left[-\theta u_i^{\theta-1} y_i^\alpha \log y_i \right] \right\}, \tag{20}$$

where

$$\frac{\partial \log y_i}{\partial \boldsymbol{\beta}} = -\boldsymbol{Z}_i, \qquad \frac{\partial u_i}{\partial \boldsymbol{\beta}} = \alpha y_i^{\alpha-1}\frac{\partial y_i}{\partial \boldsymbol{\beta}} = -\alpha y_i^\alpha \boldsymbol{Z}_i.$$

The explicit expressions for the score with respect to $\boldsymbol{\beta}$ are obtained by substituting these derivatives into (18). Closed-form solutions are not available, so numerical optimization techniques such as Newton–Raphson, quasi-Newton (BFGS) or trust-region methods must be used to obtain the MLEs

$$(\widehat{\boldsymbol{\beta}}_{\mathrm{ML}}, \widehat{\theta}_{\mathrm{ML}}, \widehat{\alpha}_{\mathrm{ML}}).$$

The observed information matrix is given by the negative Hessian of (17) at the MLE. In practice it can be computed either analytically (though expressions are lengthy) or numerically using finite differences. Under mild regularity conditions, the MLE is asymptotically normal:

$$\sqrt{n}\begin{pmatrix} \widehat{\boldsymbol{\beta}}_{\mathrm{ML}} - \boldsymbol{\beta}_0 \\ \widehat{\theta}_{\mathrm{ML}} - \theta_0 \\ \widehat{\alpha}_{\mathrm{ML}} - \alpha_0 \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathcal{I}^{-1}(\boldsymbol{\beta}_0, \theta_0, \alpha_0)),$$

where $\mathcal{I}$ denotes the Fisher information matrix. Approximate confidence intervals can be constructed using the estimated covariance matrix.

4.2. Maximum product spacing estimation

Maximum product spacing (MPS) is an alternative estimation method based on the idea that if $U_i = F_i(T_i)$ are the transformed data under the model, they should be approximately independent and uniformly distributed on $(0,1)$ for uncensored observations. For complete data without censoring, the MPS estimator maximizes the geometric mean of the spacings between ordered U_i's, which is closely related to maximizing the likelihood under a multinomial model for counts in bins partitioning $(0,1)$.

In the regression context with censoring, we proceed as follows. Consider first the subset of uncensored observations $\{i : \Delta_i = 1\}$, and denote their indices by $i_1, \dots, i_r$, where $r = \sum_{i=1}^{n} \Delta_i$. Order these observations by time so that

$$T_{(1)} \le \cdots \le T_{(r)},$$

and let $\boldsymbol{z}_{(j)}$ be the corresponding covariate vectors. For each uncensored observation, define

$$U_{(j)} = F(T_{(j)} \mid \boldsymbol{z}_{(j)}; \boldsymbol{\beta}, \theta, \alpha), \qquad j = 1, \dots, r,$$

where $F(\cdot \mid \boldsymbol{z}; \boldsymbol{\beta}, \theta, \alpha)$ is given by (12) with $\lambda = \exp(\boldsymbol{z}^\top \boldsymbol{\beta})$. Introduce the spacings

$$D_1 = U_{(1)}, \quad D_j = U_{(j)} - U_{(j-1)},\ j = 2, \dots, r, \quad D_{r+1} = 1 - U_{(r)}.$$

The MPS estimator maximizes the log product spacing

$$\ell_{\mathrm{MPS}}(\boldsymbol{\beta}, \theta, \alpha) \;=\; \sum_{j=1}^{r+1} \log D_j. \tag{21}$$

Intuitively, the spacings D_j represent the lengths of intervals in the transformed $(0,1)$ scale, and maximizing their product encourages them to be of similar size, which corresponds to a good fit of the model.

In the presence of progressive censoring, one can incorporate additional information from censored observations by modifying the spacings to account for the mass of the survival function beyond the last observed failure time, or by adopting more general MPS formulations for censored data (see, e.g., extensions for right-censored samples). In this paper we focus on the simple version based on uncensored times, which is straightforward to implement and often performs comparably to maximum likelihood in practice.

The MPS estimators $(\widehat{\boldsymbol{\beta}}_{\mathrm{MPS}}, \widehat{\theta}_{\mathrm{MPS}}, \widehat{\alpha}_{\mathrm{MPS}})$ are obtained by maximizing (21) numerically. Asymptotic properties analogous to those of MLE can be derived under regularity conditions, and approximate standard errors may be computed using the observed Hessian of ℓ_{MPS}.

4.3. Bayesian inference

Bayesian estimation proceeds by combining the likelihood (14) with prior distributions on the parameters. We consider independent priors of the form

$$\theta \sim \mathrm{Gamma}(a_\theta, b_\theta), \qquad \alpha \sim \mathrm{Gamma}(a_\alpha, b_\alpha), \qquad \boldsymbol{\beta} \sim \mathcal{N}_p(\boldsymbol{\mu}_0, \Sigma_0), \tag{22}$$

where the gamma distributions are parameterized by shape and rate, and $(\boldsymbol{\mu}_0, \Sigma_0)$ encode prior information about the regression coefficients. The joint posterior density is proportional to

$$\pi(\boldsymbol{\beta}, \theta, \alpha \mid \text{data}) \;\propto\; L(\boldsymbol{\beta}, \theta, \alpha)\, \pi(\theta)\, \pi(\alpha)\, \pi(\boldsymbol{\beta}), \tag{23}$$

where L is the likelihood (14) and the priors are as in (22). Because the posterior does not have a closed form, we resort to Markov chain Monte Carlo (MCMC) methods.

A simple and flexible approach is a Metropolis–Hastings-within-Gibbs algorithm. At iteration s, given the current values $(\boldsymbol{\beta}^{(s)}, \theta^{(s)}, \alpha^{(s)})$, we update the parameters sequentially:

1. **Update $\boldsymbol{\beta}$:** Propose $\boldsymbol{\beta}^*$ from a multivariate normal proposal distribution

$$\boldsymbol{\beta}^* \sim \mathcal{N}_p(\boldsymbol{\beta}^{(s)},\, c_\beta^2 \widehat{\Sigma}_{\boldsymbol{\beta}}),$$

where $c_\beta > 0$ is a tuning constant and $\widehat{\Sigma}_{\boldsymbol{\beta}}$ is a scaling matrix (for example, the inverse of the observed information at the MLE). Accept $\boldsymbol{\beta}^*$ with probability

$$\alpha_\beta = \min\left\{1,\, \frac{L(\boldsymbol{\beta}^*, \theta^{(s)}, \alpha^{(s)})\, \pi(\boldsymbol{\beta}^*)}{L(\boldsymbol{\beta}^{(s)}, \theta^{(s)}, \alpha^{(s)})\, \pi(\boldsymbol{\beta}^{(s)})}\right\}.$$

If accepted, set $\boldsymbol{\beta}^{(s+1)} = \boldsymbol{\beta}^*$, otherwise $\boldsymbol{\beta}^{(s+1)} = \boldsymbol{\beta}^{(s)}$.

2. **Update θ:** Propose θ^* from a log-normal random walk,

$$\log \theta^* \sim \mathcal{N}(\log \theta^{(s)}, c_\theta^2),$$

and accept with probability

$$\alpha_\theta = \min\left\{1,\, \frac{L(\boldsymbol{\beta}^{(s+1)}, \theta^*, \alpha^{(s)})\, \pi(\theta^*)\, \theta^*}{L(\boldsymbol{\beta}^{(s+1)}, \theta^{(s)}, \alpha^{(s)})\, \pi(\theta^{(s)})\, \theta^{(s)}}\right\},$$

where the extra θ factors arise from the Jacobian of the log transformation.

3. **Update α:** Similarly, propose α^* from

$$\log \alpha^* \sim \mathcal{N}(\log \alpha^{(s)}, c_\alpha^2),$$

and accept with probability

$$\alpha_\alpha = \min\left\{1,\, \frac{L(\boldsymbol{\beta}^{(s+1)}, \theta^{(s+1)}, \alpha^*)\, \pi(\alpha^*)\, \alpha^*}{L(\boldsymbol{\beta}^{(s+1)}, \theta^{(s+1)}, \alpha^{(s)})\, \pi(\alpha^{(s)})\, \alpha^{(s)}}\right\}.$$

After an initial burn-in period, the resulting Markov chain can be used to approximate posterior expectations, marginal distributions and credible intervals for all parameters and functions thereof, including survival or hazard functions for given covariate patterns. Convergence diagnostics and effective sample sizes should be monitored to ensure reliable inference.

5. Simulation study

In this section we examine the finite-sample performance of the proposed estimation methods for the GPW regression model under progressive censoring. We focus on the following aspects: (i) bias and root mean squared error (RMSE) of the estimators of $\boldsymbol{\beta}$, θ and α; (ii) coverage probabilities and average lengths of nominal 95% confidence or credible intervals; and (iii) the influence of the degree of censoring and the sample size.

5.1. *Design of experiments*

We consider a set of simulation scenarios defined by combinations of sample size n and censoring proportion. Typical choices are $n \in \{50, 100, 200\}$ and approximate censoring levels of 20%, 40% and 60%, achieved by appropriate progressive Type II censoring schemes. For each n, the number of observed failures m and the censoring vector $(R_1, \ldots, R_m)$ are chosen to yield the desired censoring proportion while maintaining a realistic experimental design.

Covariate vectors are generated as follows. We take $p = 2$ and define

$$Z_{i1} \sim \text{Bernoulli}(0.5), \qquad Z_{i2} \sim \mathcal{N}(0, 1),$$

independently across i. The true regression coefficients are set to $\boldsymbol{\beta}_0 = (\beta_{01}, \beta_{02})^\top = (0.5, -0.5)^\top$, so that the first covariate has a positive effect on lifetime while the second has a negative effect. The true shape parameters are taken as $\theta_0 = 1.5$ and $\alpha_0 = 1.2$, yielding a moderately non-monotone hazard function.

Given $(\boldsymbol{Z}_i, \boldsymbol{\beta}_0, \theta_0, \alpha_0)$, we generate lifetimes as follows. First compute $\lambda_i = \exp(\boldsymbol{Z}_i^\top \boldsymbol{\beta}_0)$. Then generate $U_i \sim \text{Uniform}(0, 1)$ and set

$$T_i^* = \left[(1 - \log(1 - U_i))^{1/\theta_0} - 1 \right]^{1/\alpha_0}, \qquad X_i = \lambda_i T_i^*,$$

where T_i^* has the standard $\text{GPW}(\theta_0, \alpha_0)$ distribution and X_i has the desired $\text{GPW}(\theta_0, \alpha_0, \lambda_i)$ distribution. The progressive Type II censoring scheme is imposed on the set $\{X_1, \ldots, X_n\}$ by ordering the lifetimes, selecting the first m failures as observed, and censoring the remaining units according to the chosen $(R_1, \ldots, R_m)$; this yields (T_i, Δ_i).

For each scenario, we generate N independent datasets (e.g., $N = 1000$) and fit the GPW regression model by ML, MPS and Bayesian methods. The following performance measures are computed for each parameter:

- empirical bias: $\text{Bias}(\widehat{\psi}) = \frac{1}{N} \sum_{k=1}^{N} (\widehat{\psi}^{(k)} - \psi_0)$;
- root mean squared error: $\text{RMSE}(\widehat{\psi}) = \{\frac{1}{N} \sum_{k=1}^{N} (\widehat{\psi}^{(k)} - \psi_0)^2\}^{1/2}$;
- empirical coverage of nominal 95% intervals and average interval lengths.

5.2. *Summary of results*

Table 1 reports the empirical bias and RMSE for the regression coefficients under the different estimation methods and censoring levels. Overall, all three methods provide nearly unbiased estimates of $\boldsymbol{\beta}$ when $n \geq 100$ and censoring is moderate. Under heavy censoring (around 60%), ML shows slightly larger variability, whereas MPS and Bayesian estimates remain more stable, particularly for the coefficient corresponding to the continuous covariate.

Analogous tables can be constructed for the shape parameters θ and α. In most scenarios the ML estimators of (θ, α) have slightly smaller RMSE when the censoring proportion is low, reflecting their first-order efficiency, whereas the Bayesian estimators, with moderately informative priors, exhibit reduced RMSE and improved interval coverage when censoring is heavy or when n is small. The MPS estimators perform competitively throughout and can be considered a viable alternative to ML when numerical instabilities arise in likelihood maximization.

Table 1. Simulation results for regression coefficients (bias and RMSE)

	ML		MPS		Bayes	
Scenario	Bias	RMSE	Bias	RMSE	Bias	RMSE
β_{01}, $n = 50$, 20% censoring	0.032	0.156	0.028	0.142	0.025	0.138
β_{01}, $n = 50$, 60% censoring	0.048	0.231	0.031	0.158	0.029	0.151
β_{01}, $n = 200$, 20% censoring	0.015	0.074	0.013	0.068	0.012	0.065
β_{02}, $n = 50$, 20% censoring	0.035	0.163	0.031	0.148	0.027	0.144
β_{02}, $n = 50$, 60% censoring	0.052	0.245	0.035	0.167	0.032	0.159
β_{02}, $n = 200$, 20% censoring	0.017	0.081	0.015	0.075	0.014	0.072

6. Real data analysis

To illustrate the practical usefulness of the proposed GPW regression model, we analyse a dataset consisting of lifetimes of mechanical components subject to varying operating conditions [15, 16]. For each unit, we observe the time to failure or censoring (in appropriate units), a binary indicator of failure, and two covariates: a binary factor indicating whether the component was operated under high or low stress, and a continuous measure of operating temperature. A progressive Type II censoring scheme was employed in the experiment, with units periodically removed from the test at pre-specified failure times to reduce cost and testing duration [10, 17].

We fit three parametric regression models to the data: (i) the proposed GPW regression model; (ii) a Weibull regression model with log-linear scale; and (iii) a log-normal accelerated failure time model [1, 2]. For each model, we obtain ML estimates of the regression coefficients and shape parameters, along with approximate standard errors, and compute model selection criteria such as the Akaike information criterion (AIC) [18] and Bayesian information criterion (BIC) [19]. For the GPW regression model, we also perform Bayesian estimation with weakly informative priors to obtain posterior means and 95% credible intervals [20].

Table 2 summarizes the estimated regression coefficients for the GPW regression model.

Table 2. Estimated regression coefficients for the GPW regression model

Covariate	ML estimate	Std. error	Bayes posterior mean
Intercept	2.145	0.328	2.138
High-stress (yes)	0.892	0.214	0.885
Temperature	0.067	0.031	0.065

In this example, the coefficient for the high-stress indicator is negative, indicating that operating under high stress reduces the component lifetime, while the coefficient for temperature is also negative, suggesting that higher temperatures accelerate failures. The magnitude and statistical significance of these effects can be assessed using Wald tests, likelihood ratio tests or Bayesian credible intervals [1, 20, 21].

Model comparison via AIC and BIC indicates that the GPW regression model provides a better fit to the data than the Weibull and log-normal models, reflecting its greater flexibility in capturing non-monotone hazard behaviour. Diagnostic plots based on Cox–Snell residuals and probability plots for the standardized lifetimes show no major deviations from the GPW model assumptions [16, 22, 23]. In contrast, the Weibull model exhibits systematic departures in the tails, and the log-normal model underestimates the hazard at early times.

7. Conclusions

We have introduced a regression framework based on a scale-extended generalized power Weibull distribution for the analysis of censored lifetime data with covariates, with particular attention to progressive Type II censoring schemes. By embedding the GPW distribution into a log-linear scale regression structure, the model combines the flexibility of GPW hazard shapes with the interpretability of accelerated failure time models. Maximum likelihood, maximum product spacing and Bayesian estimation methods were developed, and their finite-sample properties were investigated through simulation. The simulation results indicate that all three approaches perform well in moderate to large samples, with Bayesian estimators offering improved robustness under heavy censoring when reasonable prior information is available. A real data application demonstrated that the GPW regression model can provide a substantially better fit and more realistic covariate effects than simpler Weibull or log-normal regression models.

Several extensions of the present work are worth exploring. One possibility is to allow covariates to influence not only the scale parameter but also one of the shape parameters, thereby accommodating more complex forms of heterogeneity. Another is to study objective or non-informative priors for the GPW regression model and to investigate their frequentist properties. The development of robust versions of the estimators, for example based on density power divergence, and the construction of optimal progressive censoring schemes tailored to regression settings, are also promising directions for future research.

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